Three Independent Routes to $\mathrm{Fl}_{1,2}(\mathbb{C}^3) \times \mathbb{CP}^1$: Gauge Structure, Fermion Generations, and Gravity

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The product flag manifold $\mathcal{M} = \mathrm{Fl}_{1,2}(\mathbb{C}^3) \times \mathbb{CP}^1$ has been determined by the companion papers as the unique compact Kähler manifold consistent with irreducible binary recording. Here we show that three *independent* physical requirements—gauge structure (irreducible weight lattice), three generations ($\chi \equiv 0 \pmod 3$), and Einstein gravity (Bochner-flat torus quotient)—each constrain the manifold from different directions, and any two of the three are sufficient to determine $\mathcal{M}$ uniquely among compact Kähler manifolds of complex dimension 4 with effective torus action. The third requirement is then a prediction. This overdetermination—gauge theory predicts gravity, gravity predicts generations, generations predict gauge theory—is the structural signature of a framework with no free parameters.

I. INTRODUCTION

The companion papers [1–3] develop a framework in which quantum mechanics, gauge theory, and general relativity emerge from the gradient flow of a moment map component on a compact Kähler manifold $\mathcal{M} = \mathrm{Fl}_{1,2}(\mathbb{C}^3) \times \mathbb{CP}^1$. The manifold is determined by two axioms [2]: gradient flow on compact Kähler (Axiom 1) and irreducible binary recording on $\mathbb{CP}^1$ (Axiom 2).

This note makes a simple but consequential observation. The determination of $\mathcal{M}$ can be decomposed into three independent physical requirements, each of which constrains the manifold from a different direction:

G. Gauge structure: the tangent weight lattice is irreducible.

F. Three families: the Euler characteristic satisfies $\chi \equiv 0 \pmod 3$.

E. Einstein gravity: the Marsden–Weinstein quotient by the maximal torus is Bochner-flat.

We prove that any two of these three conditions determine $\mathcal{M}$ uniquely. The third is then a prediction of the other two.

II. THE CANDIDATE SPACE

We work within the class of compact Kähler manifolds M_{int} of complex dimension 3 (the internal factor; the $\mathbb{CP}^1$ recording factor is fixed by Axiom 2) with effective Hamiltonian torus action and isolated fixed points. The standard classification [2] yields five candidates:

Candidate	χ	Weights	Quotient
$\mathbb{CP}^3$	4	reducible	point
$(\mathbb{CP}^1)^3$	8	reducible	Σ
$\mathrm{Fl}_{1,2}(\mathbb{C}^3)$	6	irreducible	$\mathbb{CP}^1$
$\mathbb{CP}^1 \times \mathbb{CP}^2$	6	reducible	$\mathbb{CP}^1$
Blow-ups	$\chi + 1$	varies	varies

The "Quotient" column gives the Marsden–Weinstein quotient of M_{int} by its maximal torus at a generic regular value; the full quotient of $\mathcal{M} = M_{\mathrm{int}} \times \mathbb{CP}^1$ is therefore Quotient $\times\, \mathbb{CP}^1$.

III. THE THREE CONDITIONS

A. Condition G: Irreducible weight lattice

Proposition 1 ([2])**.** *Among the five candidates, only* $\mathrm{Fl}_{1,2}(\mathbb{C}^3)$ *has an irreducible tangent weight lattice.*

Proof. $\mathbb{CP}^3$: the torus T^3 acts with weights in $\mathbb{Z}^3$; the lattice decomposes along coordinate axes. $(\mathbb{CP}^1)^3$: manifestly decomposable. $\mathbb{CP}^1 \times \mathbb{CP}^2$: the lattice splits as $\mathbb{Z} \oplus \mathbb{Z}^2$. Blow-ups inherit the decomposability of their parent. Only $\mathrm{Fl}_{1,2}(\mathbb{C}^3)$ has weights $\alpha_1 = (1,-1)$, $\alpha_2 = (0,1)$, $\alpha_1 + \alpha_2 = (1,0)$ with $\langle \alpha_1, \alpha_2 \rangle = -1 \neq 0$ in the Killing form; no orthogonal partition exists [2]. $\square$

B. Condition F: Three families

Proposition 2 ([2])**.** *Among the five candidates, only* $\mathrm{Fl}_{1,2}(\mathbb{C}^3)$ *and* $\mathbb{CP}^1 \times \mathbb{CP}^2$ *satisfy* $\chi \equiv 0 \pmod 3$.

Proof. $\chi(\mathbb{CP}^3) = 4 \equiv 1$. $\chi((\mathbb{CP}^1)^3) = 8 \equiv 2$. $\chi(\mathrm{Fl}_{1,2}(\mathbb{C}^3)) = 6 \equiv 0$. $\chi(\mathbb{CP}^1 \times \mathbb{CP}^2) = 6 \equiv 0$. Blow-ups add one fixed point ($\chi \to \chi + 1$); none of the base cases shifted by $+1$ satisfy $\chi \equiv 0 \pmod 3$. $\square$

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C. Condition E: Bochner-flat quotient

Proposition 3 ([3]). *Among the five candidates (crossed with* $\mathbb{CP}^1$*), only* $\mathrm{Fl}_{1,2}(\mathbb{C}^3) \times \mathbb{CP}^1$ *and* $\mathbb{CP}^1 \times \mathbb{CP}^2 \times \mathbb{CP}^1$ *have Bochner-flat Marsden–Weinstein quotients.*

Proof. $\mathbb{CP}^3$: the T^3 quotient is a point; no metric dynamics. $(\mathbb{CP}^1)^3$: the T^2 quotient of the internal factor is a Riemann surface Σ; $\Sigma \times \mathbb{CP}^1$ is Bochner-flat only if Σ has constant curvature, which fails generically.

$\mathrm{Fl}_{1,2}(\mathbb{C}^3)$: the T^2 quotient is $\mathbb{CP}^1$ (coadjoint orbit reduction). The full quotient $\mathbb{CP}^1 \times \mathbb{CP}^1$ is Bochner-flat by Bryant's classification [5, 6]: a product of constant-curvature surfaces.

$\mathbb{CP}^1 \times \mathbb{CP}^2$: the T^2 quotient of $\mathbb{CP}^2$ is a point; the full quotient is $\mathbb{CP}^1 \times \mathbb{CP}^1$, Bochner-flat by the same argument.

Blow-ups introduce exceptional divisors that generically destroy the product structure and violate Bochner-flatness. □

IV. OVERDETERMINATION

Theorem 1 (Three routes). *Among compact Kähler threefolds with effective torus action and isolated fixed points, crossed with* $\mathbb{CP}^1$*:*

(i) *Conditions* ***G*** + ***F*** *determine* $\mathrm{Fl}_{1,2}(\mathbb{C}^3)$ *uniquely, and* ***E*** *is a prediction.*

(ii) *Conditions* ***G*** + ***E*** *determine* $\mathrm{Fl}_{1,2}(\mathbb{C}^3)$ *uniquely, and* ***F*** *is a prediction.*

(iii) *Conditions* ***F*** + ***E*** *determine* $\mathrm{Fl}_{1,2}(\mathbb{C}^3)$ *uniquely, and* ***G*** *is a prediction.*

Proof. (i) G eliminates all candidates except $\mathrm{Fl}_{1,2}(\mathbb{C}^3)$ (Proposition 1). F is automatically satisfied ($\chi = 6$). E is then a prediction: the quotient is $\mathbb{CP}^1 \times \mathbb{CP}^1$, which is Bochner-flat.

(ii) G eliminates all except $\mathrm{Fl}_{1,2}(\mathbb{C}^3)$. E is automatically satisfied. F is then predicted ($\chi = 6 \equiv 0$).

(iii) F retains $\mathrm{Fl}_{1,2}(\mathbb{C}^3)$ and $\mathbb{CP}^1 \times \mathbb{CP}^2$. E retains both (Proposition 3). But $\mathbb{CP}^1 \times \mathbb{CP}^2$ has a decomposable weight lattice—the recording event factors into independent sub-recordings, violating the binary atomicity of Axiom 2. Only $\mathrm{Fl}_{1,2}(\mathbb{C}^3)$ survives. G is then a prediction. □

The physical content is:

- **From G + F:** gauge theory and three generations predict Einstein gravity.
- **From G + E:** gauge theory and gravity predict three generations.
- **From F + E:** three generations and gravity predict the irreducible gauge structure of SU(3).

Each pair of requirements, drawn from different sectors of physics, uniquely determines the manifold—and the third sector follows as a consequence.

V. DISCUSSION

The overdetermination proved above is the structural analogue of parameter-free prediction. In a framework with free parameters, the manifold would need to be tuned separately for each physical sector. Here, the three sectors constrain the same geometric object from independent directions. The fact that these constraints are mutually consistent is nontrivial; it would be falsified by any pair of conditions whose intersection were empty. This structure is analogous to anomaly cancellation in gauge theory, where consistency conditions across sectors restrict the theory to a unique solution.

The result clarifies the Einstein derivation in [3]: the Bochner-flatness of the quadratic quotient is not an additional assumption but a *consequence* of the gauge and generational structure that determined the manifold. Together with the truncation hierarchy [4] and the uniqueness theorem [2], this completes a self-consistent picture: the manifold is unique, the physics at each depth is unique, and no parameter is free.

The overdetermination also sharpens the falsifiability of the framework. If any future classification of compact Kähler manifolds with torus action were to produce a sixth candidate satisfying two of the three conditions but not the third, the consistency triangle would collapse. Conversely, the survival of the triangle under any enlargement of the candidate space would constitute nontrivial evidence for the geometric origin of physical law.

[1] H. D. Kirk III, "Unified Dynamics from the Product Flag Manifold $\mathrm{Fl}_{1,2}(\mathbb{C}^3) \times \mathbb{CP}^1$," submitted to Phys. Rev. D (2026), DOI: 10.5281/zenodo.18662524.

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[3] H. D. Kirk III, "The Einstein Field Equations from Bochner-Flat Kähler Reduction on $\mathrm{Fl}_{1,2}(\mathbb{C}^3) \times \mathbb{CP}^1$," Zenodo (2026), DOI: 10.5281/zenodo.19241665.

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